

Capital Allocation between the Risky and the Risk-free Asset and Optimal Risky Portfolios

Topic 3

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Main Points

- **A top-down analysis of portfolio construction includes**
 - **Capital allocation decision:** choose between risk-free asset and risky asset
 - **Asset allocation decision:** choose the proportion among risky investments (stocks, bonds, real estate, options...)
 - **Security selection decision:** choose individual securities within each asset class



Main Points

- Chapter 6 studies the capital allocation between risk-free asset vs. risky assets by looking at their risk-return trade off (technical part) and degree of risk aversion (personal part)
- Chapter 7 examines how the composition of the risky portfolio may best be determined



Capital Allocation Across Risky and Risk-Free Portfolios

- Control risk
 - Capital allocation choice
 - Fraction of the portfolio invested in Treasury bills or other safe money market securities

The Risky Asset Example

Total portfolio value = \$300,000
Risk-free value = 90,000
Risky (Vanguard & Fidelity) = 210,000
Vanguard (V) = 54%
Fidelity (F) = 46%

The Risky Asset Example Continued

$$y = \frac{210,000}{300,000} = 0.7(\text{risky assets, portfolio } P)$$

$$1 - y = \frac{90,000}{300,000} = 0.3(\text{risk-free assets})$$

Vanguard	$113,400/300,000 = 0.378$
<u>Fidelity</u>	<u>$96,600/300,000 = 0.322$</u>
Portfolio P	$210,000/300,000 = 0.700$
<u>Risk-Free Assets F</u>	<u>$90,000/300,000 = 0.300$</u>
Portfolio C	$300,000/300,000 = 1.000$

The Risk-Free Asset

- Only the government can issue default-free bonds
- T-bills are viewed as the risk-free asset

Allocating Capital: Risk-free & Risky Assets

- It's an important decision to split investment funds between safe and risky assets
- Risk free asset: T-bills (treasury bills)
- Risky assets: consisting of two mutual funds, one invested in stocks and the other invested in long-term bonds

Example

$$r_f = 7\%$$

$$\sigma_{rf} = 0\%$$

$$E(r_p) = 15\%$$

$$\sigma_p = 22\%$$

$$y = \% \text{ in } p$$

$$(1-y) = \% \text{ in } r_f$$

Expected Return of Portfolio

$$E(r_c) = yE(r_p) + (1 - y)r_f$$

r_c = complete or combined portfolio

For example, $y = .75$

$$\begin{aligned} E(r_c) &= .75(15) + .25(7) \\ &= 13\% \end{aligned}$$

Expected Return of Portfolio

- The return also equals

$$\begin{aligned} E(r_c) &= yE(r_p) + (1 - y)r_f \\ &= r_f + y[E(r_p) - r_f] \end{aligned}$$

- The base rate of return is risk-free rate, and portfolio is expected to earn a risk premium that depends on the *risk premium of the risky portfolio* and the *investor's position in the risky asset*

Variance of Portfolio

Since $\sigma_{rf} = 0$, then

$$\sigma_c = y \sigma_p^*$$

* Rule 4 in Topic 1

Variance of Portfolio

If $y = .75$, then

$$\sigma_c = .75(.22) = .165 \text{ or } 16.5\%$$

If $y = 1$

$$\sigma_c = 1(.22) = .22 \text{ or } 22\%$$

If $y = 0$

$$\sigma_c = 0(.22) = .00 \text{ or } 0\%$$

Risk & Return of Portfolio

- Risk and return of portfolio

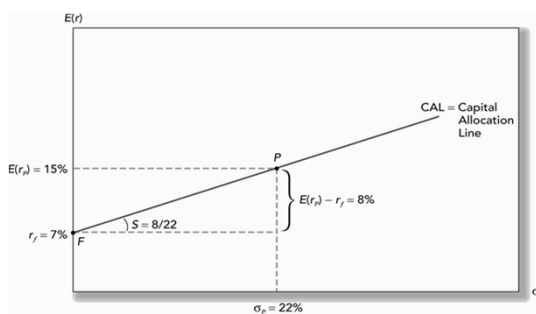
$$E(r_c) = r_f + y[E(r_p) - r_f]$$

$$\sigma_c = y \sigma_p$$

- The combination (your position) depends on the choice of y
- The slope of the capital allocation line (extra return per extra risk, reward-to-variability ratio) is

$$[E(r_p) - r_f] / \sigma_p = 8 / 22 = .36$$

CAL (Capital Allocation Line)



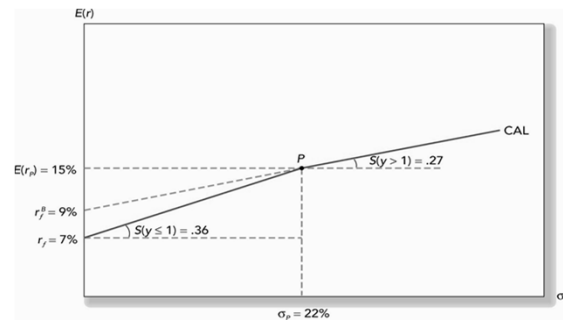
Combinations with Leverage

- Suppose the original investment is 300,000. Borrow additional \$120,000 at the risk-free rate and invest all in portfolio
- $y = 420,000 / 300,000 = 1.4$; $1 - y = -0.4$
- $r_c = .07 + 1.4 * (.08) = 18.2\%$
- $\sigma_c = (1.4) (.22) = .308 = 30.8\%$
- $S = (18.2 - 7) / 30.8 = (15 - 7) / 22 = .36$

CAL with Higher Borrowing Rate

- Non-government investors can't borrow at the risk-free rate. The borrowing rate will be higher than the lending rate, e.g. 9%
- Then $S = (15 - 9) / 22 = 0.27$
- CAL will be kinked at point P

CAL with Higher Borrowing Rate



Risk Aversion and Allocation

- After developing CAL, we need to choose one complete portfolio, C
- Risk and return trade off determines CAL
- Investor's risk aversion determines C
 - Greater levels of risk aversion lead to larger proportions of the risk free asset
 - Lower levels of risk aversion lead to larger proportions of the risky assets

Risk Aversion and Allocation

- To find C, we need three equations:

$$E(r_c) = r_f + y[E(r_p) - r_f]$$

$$\sigma_c^2 = y^2 \sigma_p^2$$

$$U = E(r) - .005 A \sigma^2$$
- To find C, solve the utility maximization problem

$$Max U = r_f + y[E(r_p) - r_f] - .005 A y^2 \sigma_p^2$$

$$y^* = \frac{E(r_p) - r_f}{.01 A \sigma_p^2}$$

Risk Aversion and Allocation

- In our example, for an investor with risk aversion of 4

$$y^* = \frac{15 - 7}{.01 * 4 * 22^2} = .41$$

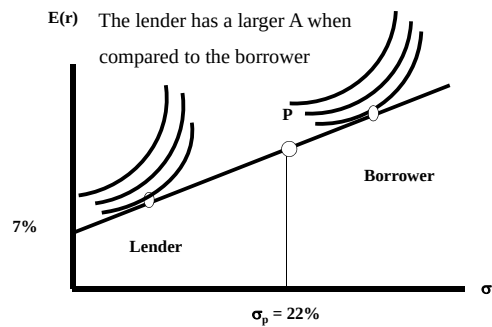
- It means that the complete portfolio is to invest 41% in risky assets and 59% in risk-free asset
- Return, risk & utility of portfolio

$$E(r_c) = 7 + [.41 * (15 - 7)] = 10.28\%$$

$$\sigma_c = .41 * .22 = 9.02\%$$

$$U = 10.28 - .005 * 4 * 9.02^2 = 8.653$$

CAL with Risk Preferences



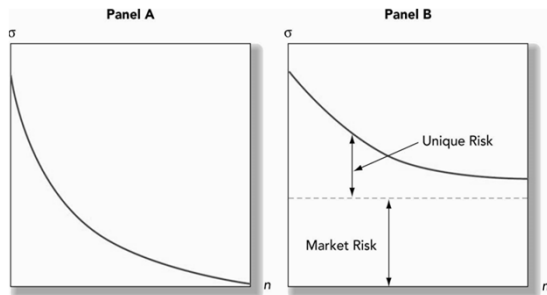
Passive Strategies

- A passive strategy avoids any direct or indirect security analysis; it allocates funds between T bills and a fund of common stocks that mimics a broad market index, i.e. S&P 500 (Capital Market Line), due to risk-return features and investor's risk aversion
- Benefit of a passive strategy
 - Low administrative and transaction costs
 - Free-rider benefit, since most assets are fairly priced

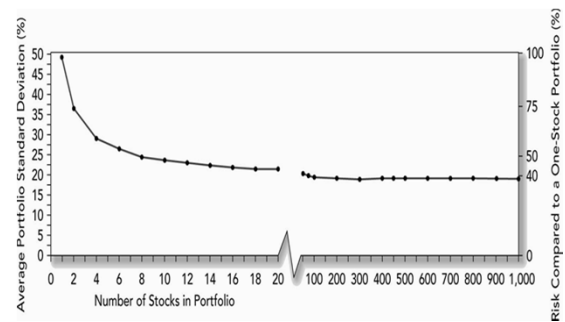
Diversification

- There are two sources of risk in a portfolio
 - Market risk (systematic risk, nondiversifiable risk)
 - Unique risk (firm-specific risk, nonsystematic risk, or diversifiable risk)
- Diversification can reduce unique risk in a portfolio, since securities don't move into the same direction

Diversification



Portfolio Diversification



Covariance and Correlation

- Portfolio risk depends on the correlation between the returns of the assets in the portfolio
- Covariance and the correlation coefficient provide a measure of the way returns of two assets vary

Diversification

- Naïve diversification uses equally weighted portfolios of several securities
- Efficient diversification constructs risky portfolios to provide the lowest possible risk for any given level of expected return
- Chapter 6 assumes risky assets as given; Chapter 7 studies how to construct the risky portfolio through efficient diversification

Two-security Portfolio: Return

$$\bar{r}_p = W_1 \bar{r}_1 + W_2 \bar{r}_2 \quad (1)$$

W_1 = Proportion of funds in Security 1
 W_2 = Proportion of funds in Security 2
 $\bar{r}_1$ = Expected return on Security 1
 $\bar{r}_2$ = Expected return on Security 2

$$\sum_{i=1}^n W_i = 1$$

Two-security Portfolio: Risk

$$\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2W_1 W_2 \text{Cov}(r_1, r_2) \quad (2)$$

Variance is not a weighted average of the individual asset variances.

$$\sigma_1^2 = \text{Variance of security 1}$$

$$\sigma_2^2 = \text{Variance of security 2}$$

$$\text{Cov}(r_1, r_2) = \text{Covariance of returns for security 1 and security 2}$$

Three-Security Portfolio(for your interest)

$$E(r_p) = w_1 E(r_1) + w_2 E(r_2) + w_3 E(r_3)$$

$$\begin{aligned} \sigma_p^2 = & w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + w_3^2 \sigma_3^2 \\ & + 2w_1 w_2 \text{Cov}(r_1, r_2) \\ & + 2w_1 w_3 \text{Cov}(r_1, r_3) \\ & + 2w_2 w_3 \text{Cov}(r_2, r_3) \end{aligned}$$

Two-security Portfolio: Risk

- The covariance of a variable with itself is the variance of that variable:

$$\begin{aligned} \text{Cov}(r_D, r_D) &= \sum_{\text{scenarios}} \text{Pr}(\text{scenario}) [r_D - E(r_D)] [r_D - E(r_D)] \\ &= \sum_{\text{scenarios}} \text{Pr}(\text{scenario}) [r_D - E(r_D)]^2 \\ &= \sigma_D^2 \end{aligned}$$

Two-security Portfolio: Risk

- Therefore, another way to write the variance of the portfolio is as follows:

$$\sigma_p^2 = W_D W_D \text{Cov}(r_D, r_D) + W_E W_E \text{Cov}(r_E, r_E) + 2W_D W_E \text{Cov}(r_D, r_E)$$

- The variance of the portfolio is a weighted sum of covariances, and each weight is the product of the portfolio proportions of the pair of assets in the covariance term

Two-security Portfolio: Risk

The variance of a two stock portfolio is the sum of these four boxes

	Stock 1	Stock 2
Stock 1	$x_1^2 \sigma_1^2$	$x_1 x_2 \sigma_{12} =$ $x_1 x_2 \rho_{12} \sigma_1 \sigma_2$
Stock 2	$x_1 x_2 \sigma_{12} =$ $x_1 x_2 \rho_{12} \sigma_1 \sigma_2$	$x_2^2 \sigma_2^2$

The variance of the portfolio is a weighted sum of covariances

Covariance

$$\text{Cov}(r_1 r_2) = \rho_{1,2} \sigma_1 \sigma_2$$

σ_1 = Standard deviation of returns for security 1

σ_2 = Standard deviation of returns for security 2

$\rho_{1,2}$ = Correlation coefficient of returns

Therefore,

$$\sigma_p^2 = w_D^2 \sigma_D^2 + w_E^2 \sigma_E^2 + 2w_D w_E \sigma_D \sigma_E \rho_{DE}$$

Correlation Effects

- Only if $\rho = +1.0$, no risk reduction is possible, portfolio S.D. is the weighted average of S.D. of two securities

$$\sigma_p^2 = (W_D \sigma_D + W_E \sigma_E)^2$$

- Otherwise, portfolio S.D. is less than the weighted average of the component S.D.

Correlation Effects

- $-1.0 \leq \rho \leq +1.0$
- The smaller the correlation, the greater the risk reduction potential
- Since correlation coefficient doesn't affect expected return, with other things equal, we always prefer to add assets with lower correlation with our existing position

Correlation Effects

- How low can portfolio S.D. be? If $\rho = -1.0$

$$\sigma_P^2 = (W_D \sigma_D - W_E \sigma_E)^2$$

$$W_D \sigma_D - W_E \sigma_E = 0$$

$$W_D = \frac{\sigma_E}{\sigma_D + \sigma_E}$$

$$W_E = \frac{\sigma_D}{\sigma_D + \sigma_E}$$

Portfolio Risk and Return

- Example with debt and equity

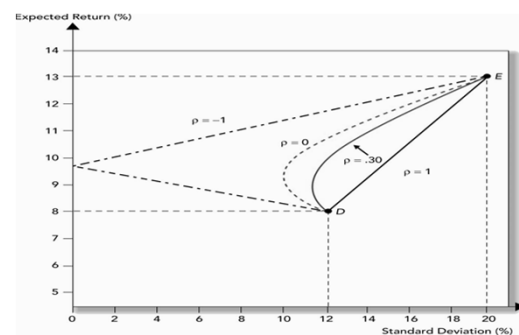
	Debt	Equity
Expected Return	8%	13%
S.D.	12%	20%
Correlation	0.3	

$$E(r_P) = 8W_D + 13W_E$$

$$\sigma_P^2 = 12^2 W_D^2 + 20^2 W_E^2 + 2 * 12 * 20 * .3 * W_D * W_E$$

- We need to find proper weights to get minimum risk portfolio (minimum-variance portfolio)

Portfolio Opportunity Sets



Portfolio Risk and Return

- Changing of W_D and W_E will change the expected return and S.D. of the portfolio
- Can W_D or $W_E > 1$ or < 0 ?
- To find the minimum level of the portfolio S.D., write out the expression for portfolio variance, substitute $1 - W_D$ for W_E , differentiate the result with respect to W_D , set the derivative equal to zero and solve for W_D

Minimum-Variance Portfolio

$$\begin{array}{lll} \text{Debt} & E(r_1) = .08 & \sigma_1 = .12 \\ \text{Equity} & E(r_2) = .13 & \sigma_2 = .20 \end{array} \quad \rho_{12} = .3$$

$$W_1 = \frac{\sigma_2^2 - \text{Cov}(r_1 r_2)}{\sigma_1^2 + \sigma_2^2 - 2\text{Cov}(r_1 r_2)}$$

$$W_2 = (1 - W_1) \quad W_1 = 0.82 \quad W_2 = 0.18$$

Minimum-Variance Portfolio

$$r_p = .82(.08) + .18(.13) = .089$$

$$\sigma_p = [(.82)^2(.12)^2 + (.18)^2(.2)^2 + 2(.82)(.18)(.3)(.12)(.2)]^{1/2}$$

$$\sigma_p = 11.45\%$$

Asset Allocation with Stocks, Bonds & Bills

- With T-bills added in, the optimal risky portfolio is the tangency point that CAL has with the investment opportunity set (the highest feasible reward-to-variability ratio)

$$Max_{W_i} S_p = \frac{E(r_p) - r_f}{\sigma_p} \quad \sum W_i = 1$$

Asset Allocation with Stocks, Bonds & Bills

- The weights of the optimal risky portfolio are: (Substitute maximization equation for $E(r_p)$ from equation (1), and for σ_p from equation (2). Substitute $1-W_D$ for W_E . Differentiate the resulting expression for S_p with respect to W_D , set the derivate equal to 0, and solve for W_D).

$$W_D = \frac{[E(r_D) - r_f]\sigma_E^2 - [E(r_E) - r_f]Cov(r_D, r_E)}{[E(r_D) - r_f]\sigma_E^2 + [E(r_E) - r_f]\sigma_D^2 - [E(r_D) - r_f + E(r_E) - r_f]Cov(r_D, r_E)}$$

$$W_E = 1 - W_D$$

$$W_D = \frac{(8-5)400 - (13-5)72}{(8-5)400 + (13-5)144 - (8-5+13-5)72} = .4$$

$$W_E = 1 - .4 = .6$$

Asset Allocation with Stocks, Bonds & Bills

- The optimal risky portfolio has 40% in debt and 60% in equity

$$E(r_p) = (.4 * 8) + (.6 * 13) = 11\%$$

$$\sigma_p = [(.4^2 * 144) + (.6^2 * 400) + (2 * .4 * .6 * 72)]^{\frac{1}{2}} = 14.2\%$$

$$S_p = \frac{11 - 5}{14.2} = .42$$

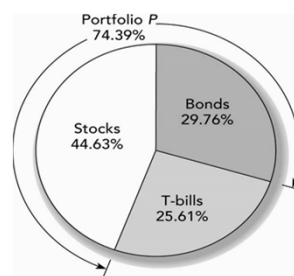
- Optimal complete portfolio is given by an optimal risky portfolio, T-bills and investor's risk aversion

$$y = \frac{E(r_p) - r_f}{.014\sigma_p^2} = \frac{11 - 5}{.01 * 4 * 14.2^2} = .7439$$

Asset Allocation with Stocks, Bonds & Bills

- So investor will invest 74.39% in optimal risky portfolio (P) and 25.61% in T-bills
- P consists of 40% in bonds, and 60% in equity, so the percentage invested in bonds is $.4 * .7439 = .2976$, and that in equity is $.6 * .7439 = .4463$

The Proportions of the Optimal Overall Portfolio



Complete Portfolio

- Specify the return characteristics of all securities (expected returns, variances, covariances)
- Establish the risky portfolio
 - Determine the weights within the optimal risky portfolio
 - Calculate the properties of P using the weights determined in previous step

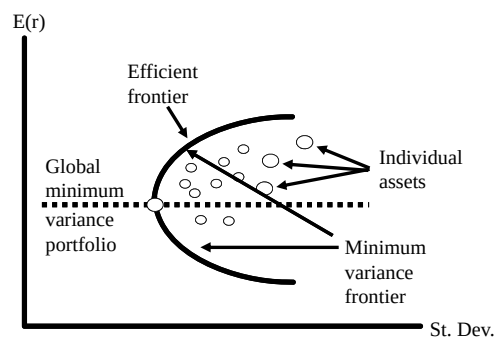
Complete Portfolio

- Allocate funds between the optimal risky portfolio and the risk-free asset
 - Calculate the fraction of the complete portfolio allocated to portfolio P and to T-bills
 - Calculate the share of the complete portfolio invested in each asset and in T-bills

Markowitz Portfolio Theory

- When there are many risky assets and a risk free asset, we have three steps to find the complete portfolio
 - First, determine the risk-return opportunities available to investors, which is summarized by minimum-variance frontier. Only the efficient frontier of risky assets (the part that lies above the global minimum-variance portfolio) is useful

Efficient Frontier



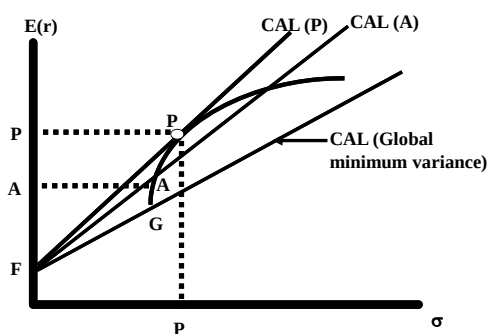
Markowitz Portfolio Theory

- Second, we draw a CAL starting from risk-free asset and tangent to the efficient frontier. The tangent portfolio P is the optimal risky portfolio with the highest reward-to-variability ratio
- Third, we choose the complete portfolio (mix between the optimal risky portfolio--P, and the risk free asset) according to individual investor's degree of risk aversion

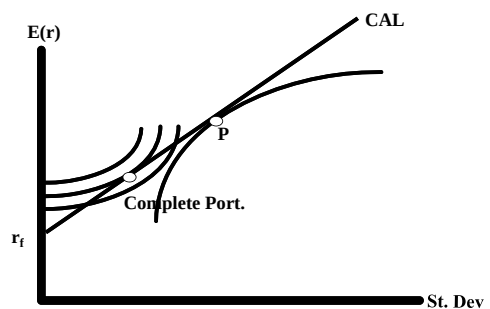
Capital Allocation and the Separation Property

- The separation property tells us that the portfolio choice problem may be separated into two independent tasks
 - Determination of the optimal risky portfolio is purely technical
 - Allocation of the complete portfolio to T-bills versus the risky portfolio depends on personal preference

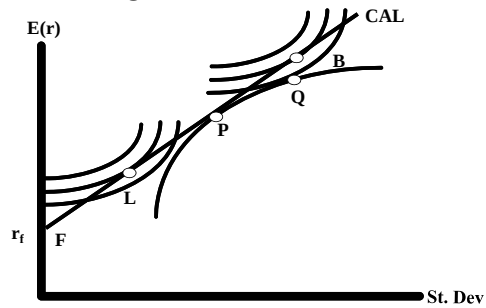
CALs



Portfolio Selection & Risk Aversion



Efficient Frontier with Lending & Borrowing



Example--1

- The change from a straight to a kinked capital allocation line is a result of the:
 - a. Reward-to-variability ratio increasing
 - b. Borrowing rate exceeding the lending rate
 - c. Investor's risk tolerance decreasing
 - d. Increase in the portfolio proportion of the risk-free asset

Example--2

- You manage an equity fund with an expected risk premium of 10% and an expected S.D. of 14%. The rate of Treasury bill is 6%. Your client chooses to invest \$60,000 of her portfolio in your equity fund and \$40,000 in a T-bill money market fund. What is the reward-to-variability ratio for the equity fund?
 - a. 0.71
 - b. 1
 - c. 1.19
 - d. 1.91

Example--3

- Which one of the following portfolios cannot lie on the efficient frontier as described by Markowitz?

	Portfolio	Expected Return%	S.D.%
a.	W	15	36
b.	X	12	15
c.	Z	5	7
d.	Y	9	21

Example--4

- Which statement is correct?
 - a. Proper diversification can reduce or eliminate systematic risk
 - b. Diversification reduces the portfolio's expected return because it reduces a portfolio's total risk
 - c. As more securities are added to a portfolio, total risk typically would be expected to fall at a decreasing rate
 - d. The risk-reducing benefits of diversification don't occur meaningfully until at least 30 individual securities are included in the portfolio

Example--5

- Portfolio theory as described by Markowitz is most concerned with:
 - a. The elimination of systematic risk
 - b. The effect of diversification on portfolio risk
 - c. The identification of unsystematic risk
 - d. Active portfolio management to enhance return

Example--6

- Stock A, B, and C have the same expected return and S.D. The following table shows the correlations between the returns on these stocks

	A	B	C
A	+1.0		
B	+0.9	+1.0	
C	+0.1	-0.4	+1.0

Given these correlations, the portfolio constructed from these stocks having the lowest risk is a portfolio:

- a. Equally invested in A&B
- b. Equally invested in A&C
- c. Equally invested in B&C
- d. Totally invested in C